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ON THE RECTILINEAR CONGRUENCE REALIZING A CIRCULAR TRANSFORMATION OF ONE PLANE INTO ANOTHER.*

By Arnold Emch.

1. In what follows I shall deal with the rectilinear (2, 2) congruence constructed by means of a circular correspondence between two planes. As far as I am aware the construction of the congruence has not previously been specialized in this way.

The general construction of congruences by means of birational correspondences, and the converse, has been studied by several authors. Steiner,† for example, established a quadratic correspondence between two planes by the intersection of the congruence of lines through any two skew-lines with those planes. Hirst,‡ on the other hand, investigated the class of congruences, based upon the Cremona transformations of two planes.

2. Consider a sphere of radius r and center M and on it any two points Z and Z' with the distance $ZZ' < 2r$ and any two planes $E \perp ZM$ and $E' \perp Z'M$, Fig. 1. E and E' may be considered as inversions of the sphere from Z and Z' as centers. Take any point P on the sphere and let A and A' be the inverses of P in the planes E and E'. Let B and B' designate the piercing-points of the straight line through Z and Z' with E and E' and let $ZZ' = s$, $ZB = p$, $Z'B' = p'$, $BB' = q$, so that $q = p + p' - s$; then

$$(1) \qquad \begin{aligned} ZP \cdot ZA &= ZZ' \cdot ZB = p \cdot s \\ Z'P \cdot Z'A' &= ZZ' \cdot Z'B' = p' \cdot s. \end{aligned}$$

If now E' and E represent complex planes, then the relation of the points A' and A is that of a bilinear, or circular transformation: §

$$(2) \qquad z' = \frac{az + b}{cz + d}.$$

3. *The line joining all pairs of corresponding points form a congruence* whose properties may be derived in the following manner: Pass a plane F through ZZ' and P, cutting the line of intersection e of E and E' in S and

* Presented to the American Mathematical Society, September, 1911.

† Steiner: Ges. Werke, vol. 1, pp. 407–439.

‡ Hirst: On Cremonian congruences. Proc. Lond. M. S., vol. 14, pp. 259–301. For further references see Sturm: Liniengeometrie, vol. 2.

§ Harkness & Morley: Introduction to Analytic Functions, pp. 42–44. Sturm: Die Lehre von den geometrischen Verwandtschaften, vol. IV, pp. 90–95.

the sphere in the circle C. In the inversions (Z, E) and (Z', E'), the straight lines SB and SB' respectively correspond to C. It is seen that the line AA' intersects the straight line through ZZ', and consequently, since P is any point on the sphere, all lines of the congruence pass through ZZ'. Letting P describe C and designating by $P, P_1, P_2, \cdots$ any set of points on

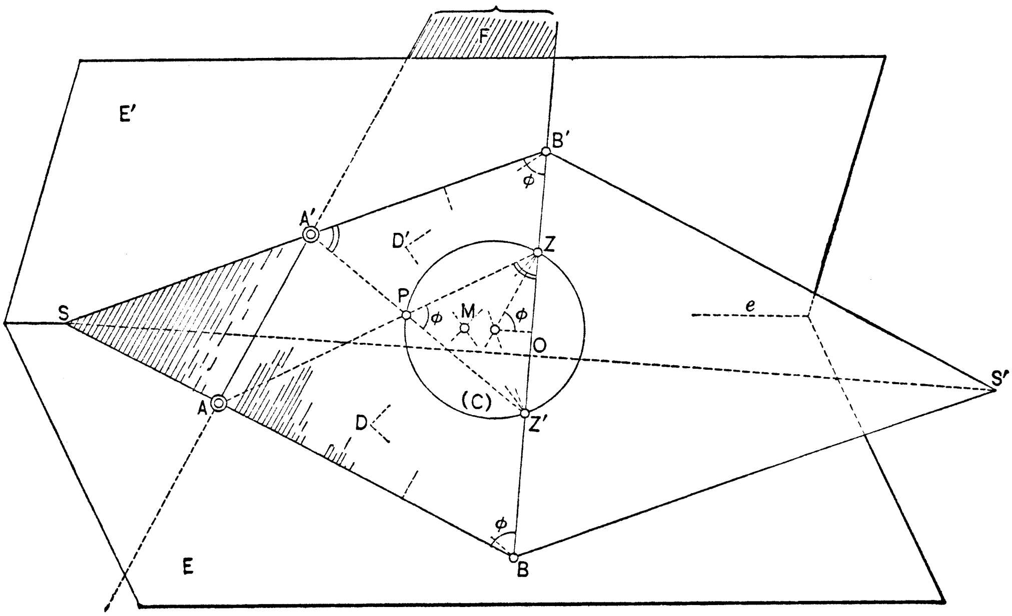

FIG. 1.

C and by $A, A_1, A_2, \cdots$ and $A', A_1', A_2', \cdots$ their projections on SB and SB' in E and E', there is plainly

$$(Z \cdot AA_1A_2 \cdots) \barwedge (Z' \cdot A'A_1'A_2' \cdots),$$

and consequently

(3) $$(AA_1A_2 \cdots) \barwedge (A'A_1'A_2' \cdots).$$

The product of these projective ranges is a conic K. We have therefore the theorem:

The envelope of all lines of the congruence situated in a plane through ZZ' is a conic.

With every point P of the sphere, not coinciding with Z or Z', is associated a definite line AA' of the congruence. If P moves on C till it coincides with Z, it is plainly seen that, since the tangent to C at Z is parallel to SB, the ray corresponding to Z is $B'S' \parallel BS$. Similarly to Z' corresponds the line $BS' \parallel B'S$. From the figure $\Delta ZAB \sim \Delta ZZ'P$ and $\Delta ZZ'P \sim$

$\Delta A'Z'B'$, hence

(4) $$\Delta ZAB \backsim \Delta A'Z'B',$$

and $\measuredangle ABZ = \measuredangle A'B'Z'$. From this follows that BB' is equally inclined towards E and E' and that $SBS'B'$ is a rhombus. The conic K is therefore always in- or escribed to a rhombus and is accordingly an ellipse or an hyperbola with the middle point O of the distance BB' as a center. When the plane F turns about BB', the variable conic K in F generates a certain surface Q of which BB' is a singular line. The congruence of lines consists now of all lines passing through BB' and tangent to Q. From any point G of BB' two tangents can be drawn in every plane through BB', namely the tangents to the conic K in F. *The lines of the congruence through any point G of BB' form therefore a cone T of the second order.* Generally the order of a tangent cone to a surface is the same as that of the surface. As the singular line is a double line, the order of the complete tangent cone from G to Q is 4, *so that the surface Q itself is of the fourth order.* To determine order and class of the congruence, choose any point in space and pass a plane through it and BB'. The lines of the congruence in this plane envelope a conic and two of its tangents, or two lines of the congruence pass through the arbitrarily chosen point. The congruence is therefore of the second order. Any plane cuts BB' in a point G and the corresponding cone T in two elements. Every plane in a general position contains therefore two lines of the congruence, and the later is therefore of the second class. Hence the theorem:

The lines joining corresponding points in a circular transformation of two planes, as defined by (2), *form a* (2, 2) *congruence. All lines of the congruence are tangent to a surface Q of the fourth order and pass through the singular line of this surface.*

4. To determine the character of the surface Q more definitely, it is necessary to go back to the origin of the conic K. The directions of the axes of all conics in- or escribed to a rhombus coincide with the diagonals of the rhombus. The lengths of the axes of K in F, Fig. 1, are determined by those positions of AA' which are perpendicular to SS' and BB'. Designating the radius of C by ρ and the angle of BB' with BS or $B'S$ by φ, then from the figure

$$\cos\varphi = \frac{\sqrt{\rho^2 - \frac{s^2}{4}}}{\rho}, \qquad SB' = \frac{q}{2\cos\varphi} = \frac{q\rho}{\sqrt{4\rho^2 - s^2}},$$

$$SO = \sqrt{SB'^2 - \frac{q^2}{4}} = \sqrt{\frac{q^2\rho^2}{4\rho^2 - s^2} - \frac{q^2}{4}} = \sqrt{\frac{q^2 s^2}{4(4\rho^2 - s^2)}} = \frac{qs}{2\sqrt{4\rho^2 - s^2}}.$$

From (4) $AB \cdot A'B' = BZ \cdot B'Z' = pp'$. When $AA' \perp SS'$, $AB = A'B'$,

hence $AB^2 = pp'$. Designating in this case the perpendicular distance from O to AA' by α, then $\alpha^2 : AB^2 = SO^2 : SB'^2$, and, introducing the above values, solving for α^2 and reducing,

$$\alpha^2 = \frac{pp's^2}{4\rho^2}. \tag{5}$$

When $AA' \perp BB'$, then $AS + SB = AB$, $SB - SA = A'B'$, hence $SB^2 - AS^2 = pp'$, and $AS^2 = SB^2 - pp'$, or

$$AS^2 = \frac{q^2\rho^2}{4\rho^2 - s^2} - pp'.$$

Designating in this case by β the perpendicular distance of O from AA', we have $\beta^2 : AS^2 = OB^2 : SB^2$, and

$$\beta^2 = \frac{q^2\rho^2 - pp'(4\rho^2 - s^2)}{4\rho^2}. \tag{6}$$

By subtraction of (5) and (6) we get

$$\alpha^2 - \beta^2 = \frac{4pp' - q^2}{4} = k^2 \text{ (a constant).} \tag{7}$$

From (5) we notice that α^2 is always a positive quantity (p and p' both positive). We can however dispose in such a manner of the points Z, Z', B, B', that in (6) β^2 may be either positive, as in the case of the figure, or negative. In the first case ($\beta^2 > 0$) K is an ellipse, in the second case ($\beta^2 < 0$) K is an hyperbola. We may also have $k = 0$, in which case all conics K are circles. Hence from (7) the result:

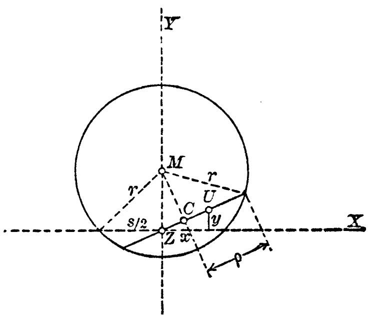

Fig. 2.

The focal distance of all conics K is constant, i. e., the foci of all conics K are situated on a circle whose plane is perpendicular to BB' and whose center is at O.

To establish the equation of the surface Q, choose BB' as the z-axis, the lines through O parallel and perpendicular to e as the x- and y-axes. Projecting the configuration on the xy-plane, Fig. 2 is obtained. The sphere appears as a circle, and the plane F with the conic K as a chord through Z. Let x, y, z be the coördinates of any point U on K (surface Q), then from Fig. 2,

$$\frac{y^2}{x^2} = \frac{CZ^2}{CM^2} = \frac{r^2 - \frac{s^2}{4} - r^2 + \rho^2}{r^2 - \rho^2}$$

and from this

$$\rho^2 = \frac{s^2x^2 + 4r^2y^2}{4(x^2 + y^2)}.$$

The equation of the surface Q is now

$$\frac{x^2 + y^2}{\alpha^2} + \frac{z^2}{\beta^2} = 1,$$

or, after replacing α^2, β^2 by their values in (5) and (6) and also ρ^2 by the above expression, and reducing,

$$(8) \quad \begin{aligned} &(x^2 + y^2)[pp's^2(s^2x^2 + 4r^2y^2) - p^2p'^2s^4] - (s^2x^2 + 4r^2y^2) \\ &\qquad [k^2(s^2x^2 + 4r^2y^2) - pp's^2z^2 - k^2pp's^2] = 0, \end{aligned}$$

which clearly is of the fourth degree.

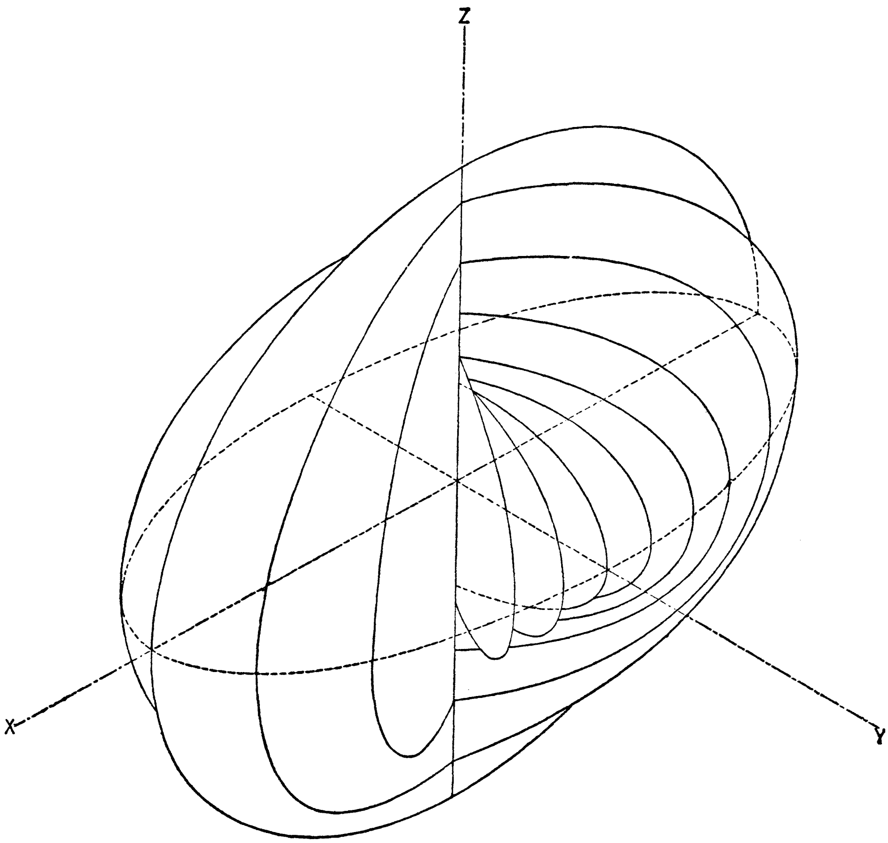

FIG. 3.

The intersection of Q with the xy-plane degenerates into the ellipse

$$\frac{x^2}{pp'} + \frac{y^2}{\frac{pp's^2}{4r^2}} = 1$$

and the point ($x = 0$, $y = 0$).

5. The peculiar character of this surface is shown in Fig. 3 by an isometric projection. In this representation $r = s = 1$, $p = 6$, $p' = 5$, $q = p + p' - s = 10$, $k^2 = 5$; and the principal axis of Q along the x-axis measures 10 units.

The equation becomes

$$(x^2 + 4y^2 - 30)(5x^2 + 2y^2) + (x^2 + 4y^2)6z^2 = 0 \tag{9}$$

under these assumptions.

6. As stated before, when $k^2 = 4pp' - q^2 = 0$, all conics K become circles. The condition is equivalent to $4pp' - (p + p' - s)^2 = 0$, or $(p - p')^2 - 2(p + p')s + s^2 = 0$. This gives for s the values $s = (\sqrt{p} \pm \sqrt{p'})^2$. The equation of Q reduces to

$$(x^2 + y^2 + z^2)(s^2x^2 + 4r^2y^2) - pp's^2(x^2 + y^2) = 0. \tag{10}$$

For $p = g$, $p' = 4$ we get as a value of s, $s = 1$, and putting $r = 1$, (10) becomes

$$(x^2 + y^2 + z^2)(x^2 + 4y^2) - 36(x^2 + y^2) = 0. \tag{11}$$

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